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On Weakly Pure Submodules of Locally Multiplication Modules

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Abstract

In this paper, the definition of $\mathcal{B}$ −weakly pure submodule is introduced and studied and also weakly pure submodules of locally multiplication modules are studied and some properties that concern this type of modules in locally multiplication modules are proved. In addition, several properties of multiplication modules have been extended to locally multiplication modules and some relations that concerning weakly pure submodules and $\mathcal{B}$ −weakly pure submodules are obtained.

1. Introduction

In 2004, M. M. Ali and D. J. Smith have been studied pure submodules of multiplication modules [10], they obtained some properties and some relations of pure submodules with some other types of submodules. In 2010, L. H. Jahromi and A. Khaksari have been studied weakly pure submodules of multiplication modules [8], which are generalizations of pure submodules and they proved several properties of this type of submodules. Also, in 2011, A. Khaksari has been studied weakly pure submodules of multiplication modules [4], and in 2014, B. N. Shihab, H. Y. Khalaf and L. S. Mahmood have been studied purely and weakly purely cancellation modules [6] and they proved some properties of each one and obtained some relations between them. In this paper we study weakly pure and $\mathcal{B}$ −weakly pure submodules of locally multiplication modules.

Throughout this paper, R is a commutative ring with identity and M is a left R −module, unless otherwise stated. Let $\emptyset \neq S \subseteq R$, then S is called a multiplicatively system if $0 \notin S$ and $a, b \in S$ implies that $ab \in S$ [7]. If S is a multiplicatively system in R, then we denote the localization of R at S by R_S (or $R^{-1}A$ [7]), which is the ring $R_S = \{\frac{r}{s} : r \in R, s \in S\}$ [7] and it is called the localization of R at S. If P is a prime ideal of R, then one can easily get that $R \setminus P$ is a multiplicatively system in R and in this case, we denote the localization of R at $R \setminus P$ by R_P, so that $R_P = \{\frac{a}{p} : a \in A, p \notin P\}$. A proper ideal A of R is called a Boolean ideal if every element of A is idempotent [4]. A proper submodule K of M is called a pure submodule of M, if $AK = K \cap AM$, for every ideal A

of R and it is called weakly pure, if $AK = K \cap AM$, for every Boolean ideal A of R [4]. M is called a multiplication R −module, if L is any submodule of M, then $L = AM$, for some ideal A of R [14] and it is called locally multiplication, if M_P is a multiplication R_P −module for each prime ideal P of R. K is called a prime submodule of M, if $r \in R, m \in M$ such that $rm \in K$, then $m \in K$ or $rM \subseteq K$ [4] and it is called weakly prime, if $r \in R, m \in M$ such that $0 \neq rm \in K$, then $m \in K$ or $rM \subseteq K$ [13] and M is called a prime module, if the zero submodule of M is a prime submodule of M [13]. If N is a submodule of M, then $S_M(N) = \{r \in R: rx \in N$, for some $x \in M \setminus N\}$ [12] and if A is an ideal of R, then $S_R(A) = \{r \in R: ra \in A$, for some $a \notin A\}$ [5]. For a submodule K of M, $(K:M) = \{r \in R: rM \subseteq K\}$ and $Ann(M) = (0:M) = \{r \in R: rM = 0\}$. M is called a finitely generated R −module if there exists a finite set $\{x_i\}_{i=1}^{n}$ in M such that $M = Rx_1 + Rx_2 + \cdots + Rx_n$, and it is called a cyclic module, if $M = Rx$, for some $x \in M$ [11]. M is called a faithful module if $Ann(M) = (0:M) = 0$ [9] and M is called a free module if it has a basis. Finally, we mention that in all sections of this paper, R is a commutative ring with identity, P is a prime ideal of R, S is a multiplicative system in R and M is an R −module.

2. Some Preserved Properties of Ideals and Submodules in Locally Multiplication Modules

In this section, we look for those properties of ideals and submodules which are preserved under localization. The first result of this section shows that under certain condition on multiplicative systems localization of Boolean Ideals are Boolean.

Theorem 2.1. Let A be a Boolean ideal of R such that $S_R(A) \cap S = \emptyset$, then A_S is a Boolean ideal of R_S. In particular, if $S_R(A) \subseteq P$, then A_P is a Boolean ideal of R_P.

Proof. If $A_S = R_S$, then by [7, Proposition 3.5], we have $A \cap S \neq \emptyset$ and as $A \subseteq S_R(A)$, we get $S_R(A) \cap S \neq \emptyset$, which is a contradiction, so that A_S is a proper ideal of R_S. Let $\frac{a}{s} \in A_S$, where $a \in R, s \in S$, then we have $qa \in A$, for some $q \in S$. If $a \notin A$, then $q \in S_R(A)$, which contradicts the fact that $S_R(A) \cap S = \emptyset$, so that $a \in A$ and then $sa \in A$. As, A is Boolean, we get $s^2a^2 = (sa)^2 = sa$ and $a^2 = a$, then $(\frac{a}{s})^2 = \frac{a^2}{s^2} = \frac{a}{s^2} = \frac{sa}{s^3} = \frac{s^2a^2}{s^3} = \frac{a^2}{s} = \frac{a}{s}$. Hence, A_S is a Boolean ideal of R_S. The proof of the second part follows directly by taking $S = R \setminus P$.

Next, we show that every submodule of a localization module at certain type of multiplicative systems is a localization of a submodule of the module itself.

Theorem 2.2. Let $\overline{N}$ be a submodule of M_S. Then $\overline{N} = N_S$, for some submodule N of M with $S_M(N) \cap S = \emptyset$. Furthermore, if $\overline{N}$ is proper, then N is proper. In particular, $\overline{N} = N_P$, with $S_M(N) \subseteq P$.

Proof. Let $s \in S$ and set $N = \{x \in M: \frac{sx}{s} \in \overline{N}\}$. It can be easily shown that N is a submodule of M. Let $\frac{x}{t} \in \overline{N}$, where $x \in M, t \in S$, then $\frac{sx}{s} = \frac{ts}{s}\frac{x}{t} \in \overline{N}$, so that $x \in N$ and thus $\frac{x}{t} \in N_S$. Hence, $\overline{N} \subseteq N_S$. Now, let $\frac{x}{t} \in N_S$, then $qx \in N$, for some $q \in S$, then $\frac{s(qx)}{s} \in \overline{N}$. Now, we have $\frac{x}{t} = \frac{s}{s}\frac{q}{q}\frac{x}{t} = \frac{1}{qt}\frac{s(qx)}{s} \in \overline{N}$, so that $N_S \subseteq \overline{N}$. Hence, $\overline{N} = N_S$. If $S_M(N) \cap S \neq \emptyset$, then there exists $p \in S$ and $p \in S_M(N)$, then $pa \in N$, for some $a \notin N$. Then, $\frac{sa}{s} = \frac{p}{p}\frac{sa}{s} = \frac{s(pa)}{ps} \in N_S = \overline{N}$, so that we get $a \in N$, which is a contradiction. Hence, $S_M(N) \cap S = \emptyset$. If $N = M$, then clearly, we have $\overline{N} = N_S = M_S$, so that $N \neq M$ and that N is proper. The proof of the second part follows directly by taking $S = R \setminus P$.

Now, by considering R as an R −module we can prove the same property of Theorem 2.2 for ideals.

Theorem 2.3. If $\bar{A}$ is an ideal of R_S, then $\bar{A} = A_S$ for some ideal A of R with $S_R(A) \cap S = \emptyset$, where $A = \{a \in R: \frac{sa}{s} \in \overline{A}\}$, for some $s \in S$. Furthermore, if $\bar{A}$ is proper, then A is proper. In particular, $\bar{A} = A_P$ with $S_R(A) \subseteq P$.

Proof. The proof follows directly by considering R as an R −module in Theorem 2.2 and for the second part take $S = R \setminus P$ in Theorem 2.2.

In the next result, we show that every Boolean ideal of a localization ring at certain type of multiplicative systems is a localization of a Boolean ideal of the ring itself.

Theorem 2.4. Let $S_R(0) \cap S = \emptyset$. If $\bar{A}$ is a Boolean ideal of R_S, then $\bar{A} = A_S$ and $S_R(A) \cap S = \emptyset$, for some Boolean ideal A of R. In particular, if $S_R(0) \subseteq P$, then $\bar{A} = A_S$ and $S_R(A) \subseteq P$.

Proof. By Theorem 2.3, we have $\bar{A} = A_S$, where $A = \{r \in R: \frac{sr}{s} \in \bar{A}\}$ is an ideal of R with $S_R(A) \cap S = \emptyset$. To show that, A is a Boolean ideal. Let, $x \in A$ be any element, then we have, $\frac{sx}{s} \in \bar{A} = A_S$, then we get $\frac{s^2x^2}{s^2} = (\frac{sx}{s})^2 = \frac{sx}{s}$, this last equation can be reduced to $t(x^2 - x) = 0$, for some $t \in S$. If $x^2 - x \neq 0$, then $t \in S_R(0)$, which contradicts the fact that $S_R(0) \cap S = \emptyset$, so that $x^2 - x = 0$, thus $x^2 = x$. Hence, A is a Boolean ideal of R. The proof of the second part follows directly by taking $S = R \setminus P$.

Next, we give a condition which makes the localization of a weakly pure submodule as weakly pure.

Theorem 2.5. If N is a weakly pure submodule of M such that $S_R(0) \cap S = \emptyset$, then N_S is a weakly pure submodule of M_S. In particular, if $S_R(0) \subseteq P$, then N_P is a weakly pure submodule of M_P.

Proof. Let $\bar{A}$ be a Boolean ideal of R_S. As, $S_R(0) \cap S = \emptyset$, by Theorem 2.4, there exists a Boolean ideal A of R such that $\bar{A} = A_S$ and since, N is a weakly pure submodule of M, so $AN = N \cap AM$. Then, we get $\bar{A}N_S = A_SN_S = (AN)_S = (N \cap AM)_S = N_S \cap (A_SM_S) = N_S \cap (\bar{A}M_S)$. Hence, N_S is a weakly pure submodule of M_S. The proof of the second part follows directly by taking $S = R \setminus P$.

It is known that, if the localization of an ideal (of a submodule) of a ring (of a module) is the zero ideal (the zero submodule), then the ideal (the submodule) may not be zero. In the following two results, we prove that under certain condition the converse of above property is also true.

Theorem 2.6. We have the following.

(1) If K, L are submodules of M such that $K_S \subseteq L_S$ and $S_M(L) \cap S = \emptyset$, then $K \subseteq L$. In particular, if $K_S = 0$ and $S_M(0) \cap S = \emptyset$, then $K = 0$.

(2) If A, B are ideals of R such that $A_S \subseteq B_S$ and $S_R(B) \cap S = \emptyset$, then $A \subseteq B$. In particular, if $A_S = 0$ and $S_R(0) \cap S = \emptyset$, then $A = 0$.

Proof. **(1)** Let $x \in K$, then for $s \in S$, we have $\frac{x}{s} \in L_S$, then $tx \in L$, for some $t \in S$. If $x \notin L$, then $t \in S_M(L)$ and as $S_M(L) \cap S = \emptyset$, we get $t \notin S$, which is a contradiction, so that $x \in L$. Hence, $K \subseteq L$. The proof of the second part follows by taking $L = 0$ in **(1)**.

(2) By using the same technique as in **(1)** and considering R as an R −module, the result follows.

Corollary 2.7. We have the following.

(1) If K, L are submodules of M such that $K_P \subseteq L_P$ and $S_M(L) \subseteq P$, then $K \subseteq L$. In particular, if $K_S = 0$ and $S_M(0) \subseteq P$, then $K = 0$.

(2) If A, B are ideals of R such that $A_P \subseteq B_P$ and $S_R(B) \subseteq P$, then $A \subseteq B$. In particular, if $A_S = 0$ and $S_R(0) \subseteq P$, then $A = 0$.

Proof. The proof follows directly by taking $S = R \setminus P$ in Theorem 2.6.

Next, we give a condition on multiplicative systems under which the localization of a free module is also free.

Theorem 2.8. Let $S_R(0) \cap S = \emptyset = S_M(0) \cap S$. If M is free, then M_S is a free R_S −module.

Proof. Let $\{m_i\}_{i=1}^{n}$ be a basis of M. Take $s \in S$. To show $\{\frac{sm_i}{s}\}_{i=1}^{n}$ is a basis for M_S. Let $\frac{x}{p} \in M_S$, for $x \in M, p \in S$, then $x = \sum_{i=1}^{n} r_i m_i$, for $r_i \in R$. Now, $\frac{x}{p} = \frac{\sum_{i=1}^{n} r_i m_i}{p} = \sum_{i=1}^{n} \frac{r_i}{p} \frac{sm_i}{s}$. Hence, $\{\frac{sm_i}{s}\}_{i=1}^{n}$ generates M_S. Next, let

$\sum_{i=1}^{n}\frac{r_i}{p_i}\frac{sm_i}{s}=0$, where $r_i\in R, p_i\in S$, for each i $(1\leq i\leq n)$, then $\sum_{i=1}^{n}\frac{r_im_i}{p_i}=\sum_{i=1}^{n}\frac{r_i}{p_i}\frac{sm_i}{s}=0$, from which we get $\frac{\sum_{i=1}^{n}(\prod_{k\neq i}^{n}p_k)r_im_i}{\prod_{i=1}^{n}p_i}=0$, so that $t\sum_{i=1}^{n}(\prod_{k\neq i}^{n}p_k)r_im_i=0$, for some $t\in S$. If $\sum_{i=1}^{n}(\prod_{k\neq i}^{n}p_k)r_im_i\neq 0$, then $t\in S_M(0)$, which contradicts the fact that $S_M(0)\cap S=\emptyset$, so that $\sum_{i=1}^{n}(\prod_{k\neq i}^{n}p_k)r_im_i=0$. As, $\{m_i\}_{i=1}^{n}$ is linearly independent, so for each i, we get $(\prod_{k\neq i}^{n}p_k)r_i=0$. Since, $S_R(0)\cap S=\emptyset$ and $\prod_{k\neq i}^{n}p_k\in S$, so we get $r_i=0$, for each i, which gives that, $\frac{r_i}{p_i}=0$, for each i, so that $\{\frac{sm_i}{s}\}_{i=1}^{n}$ is a basis for M_S. Hence, M_S is a free R_S −module.

Now, we give the following corollary to Theorem 2.8.

Corollary 2.9. Let $S_R(0), S_M(0)\subseteq P$. If M is free, then M_P is a free R_P −module.
Proof. By taking $S=R\setminus P$ in Theorem 2.8, the proof follows directly.

It is obvious that for certain types of multiplicative systems the localization of residual of submodules of a module is the same as the residual of their localizations as in below.
(i) For a submodule N of M with $S_M(N)\cap S=\emptyset$ $(S_M(N)\subseteq P)$, we have $(N:M)_S=(N_S:M_S)$ $((N:M)_P=(N_P:M_P))$.
(ii) For a submodule N of M with $S_M(N), S_R(0)\subseteq P$, we have $(Ann(N:M))_P=Ann(N_P:M_P)$ $(Ann(N_P:M_P)=(Ann(N:M))_P)$.

Theorem 2.10. If A is an ideal of R such that A_S is an idempotent ideal of R and $S_R(A^2)\cap S=\emptyset$, then A is idempotent. In particular, if A_P is idempotent in R_P and $S_R(A^2)\subseteq P$, then A is idempotent.
Proof. Let $\bar{A}$ be any idempotent ideal of R_S, then $(\bar{A})^2=\bar{A}$. By, Theorem 2.3, $\bar{A}=A_S$, for ideal $A=\{r\in R:\frac{sr}{s}\in\bar{A}\}$ of R, for $s\in S$ is any element. Then, $A_S=\bar{A}=(\bar{A})^2=(A_S)^2=(A^2)_S$. Let $a\in A$, then $\frac{sa}{s}\in\bar{A}=(A^2)_S$, then $tsa\in A^2$, for some $t\in S$. If $a\notin A^2$, then $ts\in S_R(A^2)\cap S$, which is a contradiction. Thus, $a\in A^2$, so that $A\subseteq A^2$. Hence, $A^2=A$. The proof of the second part follows directly by taking $S=R\setminus P$.

3. Weakly Pure and $\mathcal{B}$ −Weakly Pure Submodules of Locally Multiplication Modules.

This section is devoted to study weakly pure and $\mathcal{B}$ −weakly pure submodules of locally multiplication modules and determine the relations that combining these two types of submodules but first, we introduce the following definitions.

Definition 3.1. Let N be a submodule of M. Then we define $\mathcal{B}_N^M=\{A: A$ is a Boolean ideal of R, for which $AM, AN\leq M$ and $S_R(A)\cap S=\emptyset=S_M(AN)\cap S\}$.

Definition 3.2. A submodule N of M is a $\mathcal{B}$ −weakly pure submodule of M, if $AN=N\cap AM$, for every $A\in\mathcal{B}_N^M$.
The following example illustrates the above two definitions.

Example. Now, we give an example of the above definitions. Consider Z_6 as a Z_6 −module. Take the submodule $N=\{\overline{0},\overline{2},\overline{4}\}$ in Z_6 (as a Z_6 −module) and the multiplicative system $S=\{\overline{1},\overline{5}\}$ in Z_6 (as a ring). The Boolean ideals of Z_6 (as a ring) are only, $\{\overline{0}\}$ and $\{\overline{0},\overline{3}\}$. For the Boolean ideal $A=\{\overline{0}\}$, we have $AN=\{\overline{0}\}\{\overline{0},\overline{2},\overline{4}\}=\{\overline{0}\}$ and $AM=\{\overline{0}\}Z_6=\{\overline{0}\}$, which are submodules of Z_6 and we have $S_{Z_6}(A)=S_{Z_6}(\overline{0})=\{\overline{0},\overline{2},\overline{3},\overline{4}\}$ and also $S_{Z_6}(AN)=S_{Z_6}(\overline{0})=\{\overline{0},\overline{2},\overline{3},\overline{4}\}$. Clearly we have $S_{Z_6}(A)\cap S=\emptyset=S_{Z_6}(AN)\cap S$, so that $\{\overline{0}\}\in\mathcal{B}_{N=\{\overline{0},\overline{2},\overline{4}\}}^{S=\{\overline{1},\overline{5}\}}$ and clearly we have $N\cap AM=\{\overline{0},\overline{2},\overline{4}\}\cap\{\overline{0}\}=\{\overline{0}\}=AN$. Next, for the Boolean ideal $A=\{\overline{0},\overline{3}\}$, we have $AN=\{\overline{0},\overline{3}\}\{\overline{0},\overline{2},\overline{4}\}=\{\overline{0}\}$ and $AM=\{\overline{0},\overline{3}\}Z_6=\{\overline{0},\overline{3}\}$, which are submodules of Z_6 and $S_{Z_6}(A)=S_{Z_6}(\{\overline{0},\overline{3}\})=\{\overline{0},\overline{3}\}$ and also $S_{Z_6}(AN)=S_{Z_6}(\overline{0})=\{\overline{0},\overline{2},\overline{3},\overline{4}\}$ and also we have $S_{Z_6}(A)\cap S=\emptyset=S_{Z_6}(AN)\cap S$, so that $\{\overline{0},\overline{3}\}\in\mathcal{B}_{N=\{\overline{0},\overline{2},\overline{4}\}}^{S=\{\overline{1},\overline{5}\}}$ and clearly we have $N\cap AM=\{\overline{0},\overline{2},\overline{4}\}\cap\{\overline{0},\overline{3}\}Z_6=$

$\{\bar{0},\bar{2},\bar{4}\}\cap\{\bar{0},\bar{3}\}=\{\bar{0}\}=AN$. Hence, $N=\{\bar{0},\bar{2},\bar{4}\}$ is a $\mathcal{B}$ −weakly pure submodule of Z_6. By the same technique as in the above we can show that $N=\{\bar{0},\bar{3}\}$ is also a $\mathcal{B}$ −weakly pure submodule of Z_6.

Now, we prove that, the submodules which have weakly pure localizations are $\mathcal{B}$ −weakly pure.

Theorem 3.3. If N is a submodule of M such that N_S is a weakly pure submodule of M_S, then N is a $\mathcal{B}$ −weakly pure submodule of M.

Proof. Let $A\in\mathcal{B}_N^M$ so that A is a Boolean ideal of R and $S_M(AN)\cap S=\emptyset=S_R(A)\cap S$, then by Theorem 2.1, we have A_S is a Boolean ideal of R_S and since, N_S is a weakly pure submodule of M, so $A_SN_S=N_S\cap(A_SM_S)$. Then, we get $(AN)_S=A_SN_S=N_S\cap(A_SM_S)=(N\cap AM)_S$. As, $AN\subseteq N$ and $AN\subseteq AM$, so we get $AN\subseteq N\cap AM$ and as, $S_M(AN)\cap S=\emptyset$, by Theorem 2.6, we get $N\cap AM\subseteq AN$, so that $AN=N\cap AM$. Hence, N is a $\mathcal{B}$ −weakly pure submodule of M.

As a corollary for above theorem we give the following.

Corollary 3.4. If N is a submodule of M such that N_P is a weakly pure submodule of M_P, then N is a $\mathcal{B}$ −weakly pure submodule of M.

Proof. By setting $S=R\setminus P$ in Theorem 3.3, the result follows directly.

Next, we prove that prime submodules of locally multiplication modules are $\mathcal{B}$ −weakly pure.

Proposition 3.5. Let M be a locally multiplication R −module. If N is a prime submodule of M, then N is a $\mathcal{B}$ −weakly pure submodule of M.

Proof. Let N be a submodule of M. As, N is prime, it is primal, so $S_M(N)$ is a proper ideal of R, and thus $S_M(N)\subseteq P$, for some maximal ideal P of R, then M_P is a multiplication R_P −module. By [2, Proposition 2.21], we have N_P is a prime submodule of M_P, so by [4, Theorem 1], we get N_P is a weakly pure submodule of M_P. Hence, by Corollary 3.4, we get N is a $\mathcal{B}$ −weakly pure submodule of M.

We introduce the following definitions.

Definition 3.6. We define $\mathcal{B}^{J(R)}$ as the set $\mathcal{B}^{J(R)}=\{A: A \text{ is a Boolean ideal of } R \text{ such that } S_R(A)\subseteq J(R)\}$.

Definition 3.7. Let N be a submodule of M. Define, $\mathcal{W}_S=\{N: N \text{ is a weakly pure submodule of } M \text{ such that } S_M(N)\cap S=\emptyset\}$.

Ii is clear that $S=R\setminus P$ is a multiplicative system in R and we use the notation $\mathcal{W}_P$ to denote $\mathcal{W}_{R\setminus P}$ and since, $S_M(N)\cap(R\setminus P)=\emptyset$ if and only if $S_M(N)\subseteq P$, so that $\mathcal{W}_P=\{N: N \text{ is a weakly pure submodule of } M \text{ such that } S_M(N)\subseteq P\}$.

Now, we prove the following result.

Proposition 3.8. Let M be a locally multiplication R −module and $N\in\mathcal{W}_P$, for every prime ideal P of R. If M is free and $S_R(0),S_M(0)\subseteq J(R)$, then $A(N:M)=A\cap(N:M)$, for every $A\in\mathcal{B}^{J(R)}\cap\mathcal{B}_N^M$.

Proof. Let $A\in\mathcal{B}^{J(R)}\cap\mathcal{B}_N^M$, then $A\in\mathcal{B}^{J(R)}$ and $A\in\mathcal{B}_N^M$, so that A is a Boolean ideal of R, for which $AM,AN\leq M$ and $S_R(A)\cap S=\emptyset=S_M(AN)\cap S$ and $S_R(A)\subseteq J(R)$. If P is any maximal ideal of R, then M_P is a multiplication module and as, $S_R(0),S_M(0)\subseteq J(R)$, we get $S_R(0),S_M(0)\subseteq P$, so by Corollary 2.9, we have M_P is free. As, $S_R(A)\subseteq J(R)$, we get $S_R(A)\subseteq P$, so by Theorem 2.1, we get A_P is a Boolean ideal of R_P and as, $N\in\mathcal{W}_P$, we have, N is a weakly pure submodule of M and $S_M(N)\subseteq P$. As, $S_R(0)\subseteq P$, by Theorem 2.5, we get N_P is a weakly pure submodule of M_P. Hence, by [4, Theorem 2], we get $A_P(N_P:M_P)=A_P\cap(N_P:M_P)$. Next, as $S_M(N)\subseteq P$, we get $(N:M)_P=N_P:M_P$. Hence, we get $(A(N:M))_P=(A\cap(N:M))_P$ and this result is true for every maximal ideal P of R, so by [7, Proposition 3.13], we get $A(N:M)=A\cap(N:M)$.

In the remaining two results of this paper, we give some different conditions by which we prove some properties of weakly pure submodules of locally multiplication modules.

Theorem 3.9. Let R be an integral domain and M be a locally multiplication R −module. If M is faithful and cyclic and N is a proper finitely generated weakly pure submodule of M such that $S_R(0), S_M(N), S_M(0) \subseteq J(R)$, then we have the following.

(1) $(N:M)$ is an idempotent ideal of R.

(2) $N = (N:M)N$.

(3) N is an idempotent submodule of M.

(4) $Ann(N) = Ann(N:M)$.

Proof. Let P be any maximal ideal of R. Then, one can easily get that R_P is an integral domain and M_P is a multiplication R_P −module. If $N_P = M_P$, then for $x \in M$, we have $\frac{x}{1} \in N_P$ and as $S_M(N) \subseteq J(R) \subseteq P$, so that by [2, Lemma 2.1], we get $x \in N$ and thus, $N = M$, which is a contradiction. Hence, N_P is a proper submodule of M_P. Next, since, N is finitely generated, we get N_P is finitely generated. Since, $S_M(0) \subseteq J(R) \subseteq P$, by Theorem 2.5, we get N_P is a weakly pure submodule of M_P. Let, $M = Rx$, for some $x \in M$, then $M_P = (Rx)_P = R_P \frac{x}{1}$, that means, M_P is cyclic and as M is faithful, by [9, Proposition 3.14], we get M_P is faithful. Hence, we have the following.

(1) By [4, Theorem 3], we get $(N_P:M_P)$ is an idempotent ideal of R_P. By, [1, Theorem 2.21], we get $(N:M)_P = (N_P:M_P)$, so that $(N:M)_P$ is an idempotent ideal of R_P, so that $((N:M)_P)^2 = (N:M)_P$, then we get $((N:M)^2)_P = (N:M)_P$. Since, this result is true for every maximal ideal P of R, so by [7, Proposition 3.13], we get $(N:M)^2 = (N:M)$. Hence, $(N:M)$ is an idempotent ideal of R.

(2) By [4, Corollary 4], we get $N_P = (N_P:M_P)N_P$, so that $N_P = (N_P:M_P)N_P = ((N:M)N)_P$ and as this result holds for every maximal ideal P of R, by [7, Proposition 3.13], we get $N = (N:M)N$.

(3) Since, M is locally multiplication and $S_M(N) \subseteq J(R)$, so by [3, Theorem 3.2], we have $N = (N:M)M$, thus, we get $N^2 = (N:M)^2M = (N:M)M = N$. Hence, N is idempotent.

(4) By [4, Corollary 5], we have $Ann(N_P) = Ann(N_P:M_P)$. As, $S_R(0), S_M(N) \subseteq P$, we get $(Ann(N:M))_P = Ann(N_P:M_P)$ and as $S_M(0) \subseteq J(R) \subseteq P$, by [2, Proposition 2.5], we get $(Ann(N))_P = Ann(N_P)$, so that we have $(Ann(N))_P = (Ann(N:M))_P$ and this result holds for every maximal ideal P of R, so by [7, Proposition 3.13], we get $Ann(N) = Ann(N:M)$.

Proposition 3.10. Let M be a locally multiplication R −module. If M is prime and faithful, and N is a proper weakly pure submodule of M such that $S_R(0), S_M(N), S_M(0) \subseteq J(R)$, then we have the following.

(1) $(N:M)$ is an idempotent ideal of R.

(2) N is an idempotent submodule of M.

(3) $(N:M)$ is a Boolean ideal of R.

Furthermore, if $N \neq 0$, then

(4) $Ann(N:M) = 0$.

(5) N is weakly prime if and only if it is a prime submodule of M.

Proof. Let P be any maximal ideal of R, then M_P is a multiplication R_P −module. Now, let $\frac{r}{p}\frac{m}{q} = 0$, for $r \in R, m \in M, p, q \notin P$ and $\frac{m}{q} \neq 0$. Then, $trm = 0$, for some $t \notin P$ and $m \neq 0$, and as, M is a prime module, we get $trM = 0$. Next, we have $\frac{r}{p} M_P = (\frac{t}{t}\frac{r}{p} M_P) = (trM)_P = 0$. Hence, M_P is a prime R_P −module. As, M is faithful, by [9, Proposition 3.14], we get M_P is faithful. If $N_P = M_P$, then for $x \in M$, we have $\frac{x}{1} \in N_P$ and as $S_M(N) \subseteq J(R) \subseteq P$, so that by [2, Lemma 2.1], we get $x \in N$ and thus, $N = M$, which is a contradiction. Hence, N_P is a proper submodule of M_P. Since, $S_M(0) \subseteq J(R) \subseteq P$, by Theorem 2.5, we get N_P is a weakly pure submodule of M_P. Hence, by [8, Theorem 2], we have the following.

(i) $(N_P:M_P)$ is an idempotent ideal of R_P and

(ii) $(N_P{:}\,M_P)$ is a Boolean ideal of R_P.

Then, since, $S_M(N) \subseteq P$, so by the same argument as in Theorem 3.9 **(1)**, we get $(N{:}\,M)$ is an idempotent ideal of R and this proves **(1)**. Next, as M is a locally multiplication module, so by [3, Theorem 3.2], we have $N = (N{:}\,M)M$, then $N^2 = (N{:}\,M)^2M = (N{:}\,M)M = N$, so that N is idempotent and this proves **(2)**. Since, $S_M(N) \subseteq P$, so by [1, Theorem 2.21], we have $(N{:}\,M)_P = (N_P{:}\,M_P)$. Now, let $r \in (N{:}\,M)$ be any element, then we have $\frac{r}{1} \in (N{:}\,M)_P$, then, we have $\frac{r^2}{1} = (\frac{r}{1})^2 = \frac{r}{1}$, so $t(r^2 - r) = 0$, for some $t \notin P$. If $r^2 - r \neq 0$, then we get $t \in S_R(0) \subseteq J(R) \subseteq P$, which is a contradiction, so that $r^2 = r$. Hence, $(N{:}\,M)$ is a Boolean ideal of R and this proves **(3)**. If $N_P = 0$, then by [2, Lemma 2.1], we get $N = 0$, which is a contradiction, so that $N_P \neq 0$. Hence, by [8, Corollary 4], we get $Ann(N_P{:}\,M_P) = 0$. Since, $S_R(0), S_M(N) \subseteq P$, so we get $(Ann(N{:}\,M))_P = Ann(N_P{:}\,M_P) = 0$. As, $S_R(0) \subseteq P$, by [2, Lemma 2.1], we get $Ann(N{:}\,M) = 0$, this proves **(4)**. To prove **(5)**, we have $0 \neq N_P$ is a proper weakly pure submodule of the prime multiplication faithful module N_P, so by [2, Proposition 2.21] and [8, Theorem 5], we have N is weakly prime if and only if N_P is weakly prime if and only if N_P is prime if and only if N is a prime submodule of M.

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